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Geometric theory of the equivalence of Lagrangians for constrained systems

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Abstract. The Lagrangian description of a system is analysed from a geometric viewpoint in order to find a concept for equivalence of singular Lagrangians generalising that of the regular case. Geometric and gauge equivalence of singular Lagrangians are studied and we also give some conditions in which second-order differential equations exist satisfying the dynamical equation on the final constraint submanifold.

1. Introduction

One of the most important problems in classical mechanics is the so-called inverse problem, namely, in the regular case, the determination, if possible, of a function L such that the Euler-Lagrange equations corresponding to such a function are the equations of motion. In this case a closely related point is that of the 'non-uniqueness' of L. More accurately, if we take into account that the set of the equations of motion is not so relevant as the set of its solutions, we can ask whether or not there is an alternative function L' whose associated equations have the same set of solutions as those of L, or in geometric terms, that L' defines the same vector field as L. Then L and L' are said to be equivalent. We recall that the existence of non-trivial alternative Lagrange functions lead to what have been called non-Noether constants of motion (see, e.g., Giandolfi *et al* 1981, Hojman and Harleston 1981, Cariñena and Ibort 1983). Another remaining point will be the study of the different quantum systems to which non-trivially equivalent classical systems can give rise.

On the other hand, even though most textbooks on classical mechanics do not consider but regular Lagrangian systems, there is a lot of very interesting systems that are described by singular Lagrangians. In this case the Euler-Lagrange (EL) equations cannot be written in normal form because the matrix of the coefficients of the accelerations is singular and then the set of solutions of the EL equations is not well defined and it cannot be used for defining the equivalence of singular Lagrangians.

As far as we know, the problem of the equivalence of singular Lagrangians has not been studied before, except in some papers (Kalnay and Ruggeri 1973, Tello-Llanos 1984), in which the more restrictive concept of gauge equivalence of Lagrangians was analysed in the framework of Dirac's theory of constraints. The problem of the equivalence of Lagrangians seems however to be worth a deeper analysis from a geometric point of view and this will be carried out in this paper.

The paper is organised as follows. In § 2 we give a quick review of the geometrical setting for Lagrangian systems as an introduction of the notation and we propose a concept of equivalence for singular Lagrangians generalising the regular case. In § 3 we start by recalling some basic concepts and properties of the geometry of the tangent bundle which have been recently introduced by Crampin (1983) and we present three important lemmas which will be used in the proofs of the theorem of the next section. Finally, the geometrical equivalence and gauge equivalence of Lagrangians are studied in § 4 and illustrated by means of some examples in § 5.

2. Notation and basic definition

Let Q be a differentiable manifold which may be considered as the configuration space of a mechanical system and $\pi: TQ \to Q$ the tangent bundle corresponding to the velocity phase space of the system. For any real function $L \in C^\infty(TQ)$, we can define the Legendre transformation $D_L: TQ \to T^*Q$ as follows: $D_L(q, v) = (q, \mathrm{d}L_q(v))$, where $L_q: T_qQ \to R$ is given by $L_q(v) = L(q, v)$. The map D_L can be used to pull-back to TQ the canonical symplectic 2-form ω_0 defined on T^*Q and we shall obtain a closed 2-form $\omega_L = D_L^*\omega_0 \in Z^2(TQ)$. If the rank of ω_L is constant, the function L is said to be a Lagrangian function and if, besides this, ω_L is of maximal rank, L will be called regular Lagrangian. This happens iff D_L is a local diffeomorphism and in this case the map, $\hat{\omega}: \mathscr{X}(TQ) \to \Lambda^1(TQ)$, defined by contraction, i.e. $\hat{\omega}(X) = i(X)\omega$, is an isomorphism of the $C^\infty(TQ)$-linear structures. In natural coordinates ω_L is written

$$\omega_L = \frac{\partial^2 L}{\partial v^i\, \partial v^j}\, \mathrm{d}q^i \wedge \mathrm{d}v^j + \frac{\partial^2 L}{\partial v^i\, \partial q^j}\, \mathrm{d}q^i \wedge \mathrm{d}q^j$$

and it is non-degenerate iff the Hessian matrix $\partial^2 L/\partial v^i\, \partial v^j$ is regular. Moreover, in this case it is possible to define a locally Hamiltonian dynamical system (TQ, ω_L, Γ_L) by $i(\Gamma_L)\omega_L = \mathrm{d}E_L$ (i.e. $\Gamma_L = \hat{\omega}_L^{-1}(\mathrm{d}E_L)$) with E_L the energy function defined by means of the Liouville vector field Δ, $E_L = \Delta(L) - L$. The dynamics is fully contained in the dynamical vector field Γ_L and the vector field Γ_L can be shown to be a second-order differential equation (SODE) whose integral curves satisfy the Euler–Lagrange equations of motion. Two Lagrangians L_1 and L_2 are then said to be equivalent if $\Gamma_{L_1} = \Gamma_{L_2}$. But in the case of a singular Lagrangian ω_L is degenerate and hence the vector field Γ_L is not well defined. In spite of this the equation

$$\hat{\omega}_{L_1}^{-1}(\mathrm{d}E_{L_1}) = \hat{\omega}_{L_2}^{-1}(\mathrm{d}E_{L_2})$$

can still be used for defining the equivalence of both Lagrangians but with a different meaning for $\hat{\omega}_L^{-1}$, that of inverse image.

The particular case in which there exists a second-order differential equation in $\hat{\omega}^{-1}(\mathrm{d}E_L)$ will be the most important one.

We will say that two Lagrangian functions L_1 and L_2 are primary-equivalent Lagrangians (or equivalent for short) if $\hat{\omega}_{L_1}^{-1}(\mathrm{d}E_{L_1}) = \hat{\omega}_{L_2}^{-1}(\mathrm{d}E_{L_2})$. The reason for the word 'primary' will be explained later.

The concept of equivalence we have introduced is a generalisation of the definition given for regular systems and is in perfect harmony with the geometric theory developed by Gotay *et al* (1978, 1979) for dealing with pre-symplectic systems. In fact, they developed a geometric constraint algorithm for the determination of a maximal submanifold, called the final constraint submanifold, in which the dynamical equation

$i(\Gamma_L)\omega_L = \mathrm{d}E_L$ has a 'consistent' solution. The algorithm gives a decreasing sequence of submanifolds defined by $M_0 = TQ$ and for $k \geqslant 1$

$$M_k = \{m \in M_{k-1} | \exists v \in T_m(M_{k-1}), \text{ such that } \hat{\omega}_L(m)v = \mathrm{d}E_L(m)\}.$$

The limit, which is assumed to exist, is called the final constraint submanifold C and is stable under the flow of any vector field satisfying the dynamical equation $i(\Gamma)\omega_{L|C} = \mathrm{d}E_{L|C}$. Then, given two equivalent singular Lagrangians, the respective primary constraint submanifolds coincide and therefore the algorithm gives rise to the same final constraint submanifold too, as well as to the same possible dynamics. More information on the construction of the dynamics can be found in Cariñena *et al* (1985).

Another relevant case is that of time-dependent systems but it will be postponed until we have some additional machinery available.

3. Some results of the tangent bundle geometry

In a recent paper Crampin (1983) has developed the geometry of the tangent bundle of a manifold Q for dealing with Lagrangian systems from a geometrical viewpoint. The fundamental tool is a canonical tensor field, of type (1, 1); instead of Crampin's notation S, we will use V for the vertical endomorphism of TQ. V is defined by $V(u)U = (\pi_*(u)U)^V_u$ with $u \in TQ$, $U \in T_u(TQ)$ and where w^V_u is the canonical lift of the vector $w \in TQ$ to the point $u \in TQ$, i.e. $w^V_u = (\mathrm{d}/\mathrm{d}t)(u + tv)|_{t=0}$.

The expression of V in terms of natural bundle coordinates is $V = \partial/\partial v^i \otimes \mathrm{d}q^i$. It may be used for a characterisation of second-order differential equations: a vector field $\Gamma \in \mathscr{X}(TQ)$ is a SODE if and only if $V(\Gamma) = \Delta$.

Given a function $L \in C^\infty(TQ)$ the 2-form $\omega_L = D^*_L\omega_0$ can be expressed in terms of V as follows: $\omega_L = -\mathrm{d}(\mathrm{d}L \circ V)$. In fact, it is easily seen that the 1-form $\mathrm{d}L \circ V$ is the Poincaré–Cartan form θ_L (Godbillon 1969) obtained pulling-back the canonical 1-form θ_0 on T^*Q by the Legendre transformation D_L.

The properties relating the tensor field V, also called almost tangent structure (see, e.g., Gotay and Nester 1979), and the pre-symplectic structure ω_L can be summarised as follows:

Proposition 1. Let L be a Lagrangian function defined on TQ. Then

(i) $i(V(U))\omega_L = -i(U)\omega_L \circ V$ for any $U \in \mathscr{X}(TQ)$

(ii) $i(\Delta)\omega_L = -\mathrm{d}E_L \circ V$.

From both properties we see that if ω_L is regular the Γ_L is a SODE, because $i(V(\Gamma_L))\omega_L = -i(\Gamma_L)\omega_L \circ V = -\mathrm{d}E_L \circ V = i(\Delta)\omega_L$ and ω_L being regular, $V(\Gamma_L) = \Delta$; therefore Γ_L is a SODE.

Now, if α is a 1-form on Q, $\alpha \in \Lambda^1(Q)$, we shall denote $\hat{\alpha}$ the function $\hat{\alpha} \in C^\infty(TQ)$ defined by $\hat{\alpha}(u) = \alpha_{\pi(u)}(u)$. In a similar way, for any k-form $\beta \in \Lambda^k(Q)$, $\tilde{\beta}$ will denote the basic k-form on TQ given by $\tilde{\beta} = \pi^*\beta$. In particular, for any function $h \in C^\infty(Q)$, $\tilde{h}$ will denote the function $\tilde{h} = \pi^*h = h \circ \pi$.

We give without proof three lemmas containing standard results to be used later. A proof of the third can be found in a recent paper (Cariñena and Ibort 1985).

Lemma 1. A k-form $\alpha \in \Lambda^k(TQ)$ is a basic form over Q, i.e. $\alpha = \tilde{\beta}$ for some $\beta \in \Lambda^k(Q)$, iff $i_V\alpha = 0$ and $\mathscr{L}_V\alpha = 0$ for every vertical vector field.

In particular, if α is a closed k-form, it is enough for the first condition $i_V\alpha = 0$ in order for α to be a basic form.

Lemma 2. If $\alpha \in \Lambda^1(Q)$, then $\forall X \in \mathscr{X}(Q)$ we have $X^V(\hat{\alpha}) = \widetilde{\alpha(X)}$.

Lemma 3. Let Γ be a SODE on Q. For any $h \in C^\infty(Q)$ we have

$$\mathscr{L}_\Gamma \tilde{h} = \widehat{dh}.$$

4. The gauge equivalence of Lagrangians

The geometric structure associated with a Lagrangian L is given by ω_L. If we take into account that for any $\lambda \in \mathbb{R}$, $\omega_{L_1+\lambda L_2} = \omega_{L_1} + \lambda\omega_{L_2}$, two Lagrangians giving the same 2-form, that are called geometrically equivalent, will differ in a function L_0 whose 2-form ω_{L_0} vanishes identically (see, e.g., Ibort 1984). The energy associated with L_0 can however be different from zero and they would be inequivalent Lagrangians.

Theorem 1. A function $L_0 \in C^\infty(TQ)$ is such that $\omega_{L_0} \equiv 0$ if and only if there exists a closed 1-form $\alpha \in Z^1(Q)$ and a function $h \in C^\infty(Q)$ such that $L_0 = \hat{\alpha} + \tilde{h}$.

Proof. First of all we remark that if $\alpha \in Z^1(Q)$ then $\theta_{\hat{\alpha}} = \pi^*\alpha$. In fact, if $(q, v) \in TQ$ and $Y \in T_{(q,v)}(TQ)$,

$$\theta_{\hat{\alpha}}(Y)|_{(q,v)} = d\hat{\alpha} \cdot V(Y)|_{(q,v)} = d\hat{\alpha}(\pi_* Y)^V_{(q,v)} = (\pi_* Y)^V_{(q,v)}\hat{\alpha}$$

and therefore

$$\theta_{\hat{\alpha}}(Y)|_{(q,v)} = \frac{d}{dt}\hat{\alpha}(q, v + t(\pi_* Y))|_{t=0} = \alpha_q(\pi_{*(q,v)} Y)$$

which may be written $\theta_{\hat{\alpha}} = \pi^*\alpha$. Consequently $\omega_{\hat{\alpha}} = -d\theta_{\hat{\alpha}} = -d(\pi^*\alpha) = 0$. On the other hand, it is obvious that for any $h \in C^\infty(Q)$, $d\tilde{h} \circ V = 0$, because $d\tilde{h} \circ V = d(\pi^* h) \circ V = dh \circ \pi_* V = 0$.

Conversely if $\omega_{L_0} = 0$, the 1-form $\theta_{L_0} = dL_0 \circ V$ is closed. It is also invariant under vertical field $X \in \mathscr{X}^V(TQ)$; in fact for such a field $\pi_* X = 0$ and thus $dL \circ V(X) = dL(\pi_* X)^V = 0$ and from both conditions we see that $\mathscr{L}_X\theta_L = i(X)\, d\theta_L + d i(X)\theta_L = 0$.

The results of the lemmas 1 and 2 in § 3 shows that there exists $\alpha \in Z^1(Q)$ such that $dL \circ V = \pi^*\alpha = d\hat{\alpha} \circ V$. Let now f be the difference $f = L - \hat{\alpha}$. The condition $df \circ V = 0$ implies that there is a function h such that $f = \pi^* h$, which ends the proof of the theorem.

As a consequence of the preceding theorem, two Lagrangians of mechanical type $L_i = \frac{1}{2}g - V_i$ with the same kinetic energy defined by the quadratic form associated with a Riemannian metric, have associated with the same 2-form ω_L, both differing in a Lagrangian $L_0 = V_1 - V_2$.

An equivalence relation finer than the geometrical equivalence of Lagrangians can also be considered: two Lagrangians L_1 and L_2 are said to be gauge equivalent if there is a 1-form $\alpha \in Z^1(Q)$ such that $L_2 = L_1 + \hat{\alpha}$ up to a constant. In this case, if we take into account that $\Delta(\hat{\alpha}) = \hat{\alpha}\ \forall \alpha \in \Lambda^1(Q)$, we see that $E_{L_2} = E_{L_1}$ and therefore two gauge equivalent Lagrangians are not only geometrically equivalent but equivalent, too. The converse property is true for regular Lagrangians (see, e.g., Abraham and Marsden 1978) and for a more general case as explained in the following theorem (Ibort 1984).

Theorem 2. Let L_1 and L_2 be two equivalent and geometrically equivalent Lagrangians. If there is a SODE Γ such that $(i(\Gamma)\omega)_{|C} = \mathrm{d}E_{L_1|C}$ where C is the final constraint submanifold, in particular if L_1 is regular, then both Lagrangians are gauge equivalent.

Proof. By hypothesis the two Lagrangians are related by $L_2 = L_1 + \hat{\alpha} + \tilde{h}$ with $\alpha \in Z^1(Q)$, $h \in C^\infty(Q)$. Let Γ be the SODE such that $(i(\Gamma)\omega)_{|C} = \mathrm{d}E_{L_1|C}$. The energies are related by $E_{L_2} = E_{L_1} - \tilde{h}$ and the relation $(i(\Gamma)\omega)_{|C} = \mathrm{d}E_{L_i|C} (i = 1, 2)$ implies that $\Gamma(E_{L_2}) = \Gamma(E_{L_1}) = 0$ and henceforth $\Gamma(\tilde{h}) = 0$. But $\Gamma(\tilde{h}) = \widehat{\mathrm{d}h}$, according to lemma 3, because Γ is a SODE, and consequently h is (locally) constant and both Lagrangians will be gauge equivalent.

As indicated after proposition 1, when the Lagrangian L is regular any vector field $\Gamma \in \mathscr{X}(TQ)$ satisfying the dynamical equation $i(\Gamma)\omega_L = \mathrm{d}E_L$ is a SODE. On the contrary, if L is singular the condition $V(\Gamma) = \Delta$ need not follow and it is not clear how strong the condition on the existence of a SODE Γ as in theorem 2 is. We give next some conditions in which the existence of such a SODE on the final constraint submanifold can be shown and we will give a few examples in the following section.

First of all, in order to convince us of the existence of Lagrange functions for which there is no SODE in $\hat{\omega}_L^{-1}(\mathrm{d}E_L)$ we analyse the singular case of a first-order Lagrangian. Let Q be $Q = \mathbb{R}^{2n}$ and we choose a $\lambda \in \Lambda^1(\mathbb{R}^{2n})$ such that $\mathrm{d}\lambda = \Omega$ is a symplectic form on $\mathbb{R}^{2n}$. In $TQ \simeq \mathbb{R}^{4n}$ the following Lagrange function is given: $L = \hat{\lambda} - \pi^* V$, with V an arbitrary but fixed C^∞-differentiable function on $\mathbb{R}^{2n}$. A straightforward computation gives us that $\omega_L = \widetilde{\mathrm{d}\lambda}$, i.e. ω_L is written in coordinates as follows: $\omega_L = \Sigma\, (\partial_i\lambda_j - \partial_j\lambda_i)\, \mathrm{d}x^i \wedge \mathrm{d}x^j$. Consequently Ker ω_L is made up by the vertical fields and if we take into account that $E_L = \tilde{V} = \pi^* V$, it is very easy to check that the extension $\tilde{\Gamma} = (\Gamma, 0)$ to $T\mathbb{R}^{2n} \simeq \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ of the vector field $\Gamma \in \mathscr{X}((\mathbb{R}^{2n})$ defined by $\Gamma = \hat{\Omega}^{-1}(\mathrm{d}V)$ satisfies the dynamical equation $i(\tilde{\Gamma})\omega_L = \mathrm{d}E_L$. Then $\hat{\omega}_L^{-1}(\mathrm{d}E_L) = \tilde{\Gamma} + V(TQ)$ and there is no SODE in such a subset $\hat{\omega}_L^{-1}(\mathrm{d}E_L)$.

Theorem 3. Let L be a Lagrangian function such that Ker $\omega_L \cap V(TQ)$ is a sub-bundle of $T(TQ)$ and $2\dim(\mathrm{Ker}\, \omega_L \cap V(TQ)) = \dim \mathrm{Ker}\, \omega_L$. Then, there exists a SODE Γ such that $(i(\Gamma)\omega_L)_{|C} = \mathrm{d}E_{L|C}$, where C is the final constraint submanifold.

Proof. To begin with we remark that if Ker $\omega_L \cap V(TQ) = \{0\}$ the Lagrangian L is regular because for any $U \in \mathrm{Ker}\, \omega_L$, the relation $i(V(U))\omega_L = -i(U)\omega_L \circ V$ shows that $V(U) \in \mathrm{Ker}\, \omega_L \cap V(TQ) = \{0\}$ and therefore U is itself vertical and by a second application of the condition it too is zero.

We have already seen that if L is regular $\hat{\omega}_L^{-1}(\mathrm{d}E_L)$ is a SODE. If L is singular, for any Γ in $\hat{\omega}_L^{-1}(\mathrm{d}E_L)$, $i(V(\Gamma))\omega_L = i(\Delta)\omega_L$ and therefore $X = V(\Gamma) - \Delta \in \mathrm{Ker}\, \omega_L$. Both fields, $V(\Gamma)$ and Δ, are vertical fields and consequently $V(\Gamma) - \Delta \in \mathrm{Ker}\, \omega_L \cap V(TQ)$. Then there is a vector $Y \in \mathrm{Ker}\, \omega_L$ such that $V(Y) = X$ and $\Gamma - Y$ is a SODE in $\hat{\omega}_L^{-1}(\mathrm{d}E_L)$. The existence of such a vector field Y follows from the following. The kernel of the restriction of V to Ker ω_L is Ker $\omega_L \cap V(TQ)$ and therefore there is a monomorphism $\tilde{V}$: Ker ω_L/Ker $\omega_L \cap V(TQ) \to \mathrm{Ker}\, \omega_L \cap V(TQ)$ from which we can conclude that dim Ker $\omega \leqslant 2\dim(\mathrm{Ker}\, \omega_L \cap V(TQ)$. The equality sign means that V is an epimorphism and the existence of the vector Y follows. Finally we can only assure the existence of a Γ satisfying $i(\Gamma)\omega_L = \mathrm{d}E_L$ on the final constraint submanifold C and therefore the SODE only satisfies the dynamical equation on C.

5. Examples

As our first example we are going to consider the two-dimensional system proposed by Kalnay and Ruggeri (1973):

$$L = \tfrac{1}{2}(v_1^2 - q_1^2)q_2.$$

In this system the energy is given by $E_L = \frac{1}{2}(v_1^2 + q_1^2)q_2$ and the presymplectic form ω_L by $\omega_L = q_2\,\mathrm{d}q_1 \wedge \mathrm{d}v_1 + v_1\,\mathrm{d}q_1 \wedge \mathrm{d}q_2$ (we assume that the straight line $q_2 = 0$ has been removed from the configuration space). Then the primary constraint submanifold C is determined by the constraint $\phi(q_1, q_2, v_1, v_2) = q_1^2 - v_1^2 = 0$. The kernel of ω_L is generated by the fields $q_1\,\partial/\partial q_1 - v_1\,\partial/\partial v_1$ and $\partial/\partial v_2$ and therefore we find the condition of theorem 3 and there will be a SODE in C: it is given by

$$\Gamma = v_1\,\partial/\partial q_1 + v_2\,\partial/\partial q_2 - q_2^{-1}(v_1 v_2 + q_1 q_2)\,\partial/\partial v_1.$$

Another remarkable example is that of a Lagrangian of mechanical type in which the configuration space is $\mathbb{R}^n$. The Lagrangian is given by $L(x, v) = \frac{1}{2}g(v, v) - V(x)$, but the quadratic form g defining the kinetic energy is assumed to be degenerate. In this case the presymplectic form ω_L is written as $\omega_L = g_{kl}\,\mathrm{d}x^k \wedge \mathrm{d}v^l$ and therefore Ker ω_L is $2r$-dimensional, r being the corank of the quadratic form g, i.e. it is made up by vector fields $\Gamma = a^i\,\partial/\partial x^i + b^j\,\partial/\partial v^j$, with $\boldsymbol{a}$ and $\boldsymbol{b}$ in the kernel of g. Furthermore, Ker ω_L is r-dimensional and therefore there will be a SODE Γ that satisfies the dynamical equation on the final constraint submanifold.

Time-dependent Lagrangian systems can also be considered as constrained systems when the homogeneous formalism is used. Given a time-dependent Lagrangian $L(q, v, t)$ the corresponding homogeneous Lagrangian $\mathscr{L}$ is defined on the tangent bundle $T(Q\times\mathbb{R})$ of the space of events $Q\times\mathbb{R}$ as follows: $\mathscr{L}(q, t; u, w) = L(q, u/w, t)$. The energy function $E_{\mathscr{L}}$ vanishes identically as a consequence of the homogeneity of $\mathscr{L}$ in the velocities. If L_t: $TQ \to \mathbb{R}$, given by $L_t(q, v) = L(q, v, t)$, is regular for any $t \in \mathbb{R}$, it is easy to check that Ker $\omega_{\mathscr{L}}$ is two-dimensional while Ker $\omega_{\mathscr{L}} \cap V(TQ)$ is one-dimensional: a generator is the Liouville field $\bar{\Delta}$ in $T(Q\times\mathbb{R})$, i.e. $\bar{\Delta} = u^i\,\partial/\partial u^i + w\,\partial/\partial w$. As a result of theorem 3 we know that there are SODEs satisfying the dynamical equation $i(\Gamma)\omega_{\mathscr{L}} = 0$, two of such SODE differing in an element of Ker $\omega_{\mathscr{L}} \cap V(TQ)$. The dynamics is determined by considering on the hypersurface $w = 1$ (in which the parameter of the integral curves of vector fields on $T(Q\times\mathbb{R})$ coincides with the time) the uniquely defined SODE in Ker $\omega_{\mathscr{L}}$ that is tangent to the hypersurface $w = 1$. In fact, the expression in coordinates of the equations determining the integral curves of such a vector field are the EL equations for the Langrangian L. It is noteworthy that if two time-dependent Lagrangians L_1 and L_2 (such that L_{it}: $TQ \to \mathbb{R}$ is a regular function for any t) have associated homogeneous Lagrangians that are equivalent according to the definition given in § 2 for singular Lagrangians, they will rise to the same vector field if the above method is developed and it supports the definition we have given for equivalence of singular Lagrangians.

6. Final remarks

After a quick review of the theory of the equivalence of regular Lagrangians from a geometric point of view, we have proposed a concept for equivalence of singular Lagrangians, a concept that, as far as we know, had not been proposed, probably because the set of solutions for the EL equations used for defining equivalence of regular Lagrangians is not well defined when L is singular. The equivalence relation we have introduced makes use of the primary constraint submanifold (this is the reason for the adjective 'primary' in the definition of equivalence) and a slight modification of this relation will fit better to the concept of physical equivalence: two singular Lagrangians are equivalent if they give rise to the same final constraint submanifold C and every solution of the dynamical equation on C, $(i(\omega_{L_1})\Gamma)_{|C} = dE_{L_1|C}$, is too a solution of the dynamical equation for L_2.

The (pre)symplectic form ω_L defined by the Lagrangian L is an auxiliary tool in the framework of classical mechanics but it plays a distinguished role when trying to do the corresponding quantum description of the system. Then in order to have equivalent quantum systems not only the dynamical vector fields but the forms ω_L must coincide. This is the problem we have analysed in § 4 in which we have established that if there exists a SODE satisfying the dynamical equation on the final constraint submanifold the above-mentioned equivalence of Lagrangians as giving the same presymplectic system is but the well known gauge equivalence (Lévy-Leblond 1969), just as in the regular case. In the general case however, the gauge-equivalence relation seems to be better than primary plus geometric equivalence.

We have also given a theorem showing the existence of such a SODE in a particular situation, as well as some examples in which the theorem works: the homogeneous formalism for time-dependent systems is one of them and the concept of equivalence we have proposed reduces to the standard concept of 'giving the same set of solutions of the Euler–Lagrange equations' when the Lagrangians are assumed to be regular for any time t.

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